COST OPTIMAL ALLOCATION AND RATIONING IN SUPPLY CHAINS

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ABSTRACT
A supply chain with a single supplier and multiple retailers has been considered for the study. Each retailer has a unique discrete demand distribution, unique deterministic lead-time and the inventory positions of all the retailers are known to the central decision maker. All retailers have unique holding and penalty costs. Transportation costs are not considered for the model in order to avoid possible bias while allocating or rationing inventory. We consider two cases of allocation. In the first case, the supplier has limited availability of inventory stock to distribute among the retailers and in the second case, supplier has unlimited inventory in stock and like to ration it among the retailers. Unique service level constraints of each retailer, such as the minimum required non-stock-out probability during the period in the rationing and allocation problems is also included in the model. The proposed model can handle any type of discrete demand distribution of each retailer. An iterative procedure has been proposed for solving the problem. An illustrative example is discussed.

KEY WORDS
Supply chain management, rationing, allocation, optimization, non-stock-out probability

1. INTRODUCTION
This paper focuses on allocation and rationing problems in supply chains. The objective is to cost optimally allocate the available (limited/unlimited) inventory to a number of lower level locations. Many deterministic as well as stochastic variables are to be considered when trying to solve allocation and rationing problems. Some of these variables are: demand, present stock and safety stock at each lower location, lead times, different cost elements, service levels requirements, number of locations, and available inventory for distribution. There may be more variables depending on the system and assumptions. The rest of the paper has been organized as follows. A brief review of literature of the most important recent publications on rationing and allocation problems has been discussed in section 2. The problem under investigation has been clearly stated and discussed in section 3. The proposed methodology for modeling and solving the stated problem has been discussed in section 4. A numerical example has been included in section 5 for illustrating the methodology. Major findings and conclusions are given in section 6 and the latest references are given in section 7 of the paper.

2. REVIEW OF LITERATURE
Federgruen and Zipkin (1984a) present a method to approximate a problem with several retail outlets by a single-outlet model. They obtain a near-optimal policy and a good approximation of the cost of the system. Their results suggest that the balance assumption, and hence the Clark (1960) approach, is inappropriate when coefficient of variation are unequal. Federgruen and Zipkin (1984b) address the combined problem of allocating scarce inventory among several locations and planning deliveries using a fleet of vehicles. Demands are random, holding, shortage and transportation costs are considered in the model. They extend some available deterministic methods of vehicle routing problem to this case. Federgruen and Zipkin (1983) present methods for solving allocation problems that can be stated as convex knapsack problems with generalized upper-bounds. Such bounds may express upper limits on the total amount allocated to each of the several locations. They do not consider any service level constraint in the allocation models. Eppen and Schrage (1981) analyze a depot-warehouse system with independent, normally-distributed, stationery warehouse demand; identical proportional costs of holding and backordering; and no transshipment.
Ballou (1999) explains a method for allocation of inventory to stocking points on the basis of the average demand rate, that is, the forecasted demand rate. Costs are not considered for the allocation. Diks and Kok (1999) consider a cost function (holding + penalty costs) in allocation. The allocation is based on the balanced position assumption. This allocation need not be optimum in the sense of cost minimization. Verrijdt and Kok (1995) instead of defining a cost structure, apply a service level approach where the main goal is to realize pre determined service levels in the final stock points. Chung, Flynn, and Staliski (2001) propose an allocation technique for a serial supply chain to maximize the expected profit. Diks, Kok, and Lagodimos (1996) discuss Fair Share (FS), Appropriate Share (AS), and Consistent Appropriate Share (CAS) Rationing Policies, which are used in push inventory systems. Priority Rationing (PR) is a policy used in Pull inventory system. They do not consider cost optimization, but concentrate on service levels such as non-stock-out probability, fill rate etc. Cachon and Lariviere (1999) discuss a method of mapping from retailer orders to capacity assignments. The authors show that a broad class of mechanisms is prone to manipulation. This will lead to demand amplification, popularly known as bullwhip effect. Vericourt et.al. (2002) and Ha (2000) employ queuing based approach and the models follow a single-server, single-product, make-to-stock queue with multiple demand classes. Ha (1997) considers the production control and stock rationing in a make-to-stock production system with two priority customer classes and back ordering. Kukreja et.al.(2001) propose a model to find the inventory stocking levels at all locations that minimize the sum of inventory holding costs and transshipment costs subject to a specified minimum system-service level, in a N-location continuous review system. Axsater et.al.(2002) propose a two-step allocation heuristic rule for allocating stock from warehouse to the retailers. You (2003) propose a dynamic rationing policy that decides whether or not to accept each request from a customer for any combination of demand class, remaining decision period, and remaining stock.

A critical analysis of the available literature reveals that even for two-stage systems, there is no well-demonstrated procedure, which is widely acceptable and can be easily and effectively used for general purpose rationing. Most of the related works assume various conditions, and solve the problem using intuitive, heuristic, or other approximate methods. Therefore, it appears that there is a need for developing an effective, easy-to-use, and practical methodology for solving allocation and rationing problems in supply chains. This research work is an attempt in that direction. We propose to use the basic principles of statistical methods and an iterative procedure to formulate and solve the problem.

3. PROBLEM DEFINITION
There is one supplier and N retailers. Two cases of inventory distribution have been considered. In the first case, the supplier has only limited quantity of inventory for distribution among the retailers. There are unique deterministic lead times for reaching material from the supplier to each retailer. All retailers have a policy of safety stocking to meet demand during the lead times. A central decision maker has the real time information about inventory and the demand pattern at each retailer and based on this the available inventory is allocated periodically. Each retailer has a unique discrete demand distribution, holding cost, and shortage cost. The problem is optimal rationing of available inventory with the supplier to all the retailers. The objective function is the total holding and stock-out costs of all retailers, and the constraint is the available stock and the service levels. In the second case, the supplier has unlimited inventory in stock to distribute among the retailers.

4. PROPOSED METHODOLOGY
The expected cost function of each retailer is the sum of expected holding cost and expected out-of-stock (penalty) cost of each retailer. Functions for expected number of in-stock (unsold) inventory at the end of period and expected number of out-of-stock (back order) inventory at the end of the period are derived from the unique discrete demand distribution of each retailer. The total cost function for each retailer is derived from this. The optimization problem cannot be solved by usual analytical methods, because the objective function consists of infinite number of terms. Therefore, it is solved using an iterative algorithm discussed in more details in subsequent sections. For the rationing problem, the iteration starts with assigning zero inventories for each retailer. Inventory level is increased from this
position in subsequent iterations so as to maximize the decrease of total cost compared with the total cost of the previous allocation. The procedure continues in this way till the constraint of available stock with supplier is satisfied. When all the retailers are excluded or the available inventory is fully allocated, the iteration stops. If the available inventory with the supplier is unlimited, the iteration procedure is continued till the total cost function starts increasing. The allocation matrix in the just previous iteration is the optimum allocation in this case.

**Notations:**

- \( A \) : Available stock of inventory at the Supplier
- \( \lambda \) : Mean of the Poisson demand at retailer \( i \)
- \( L_i \) : Lead time for retailer \( i \)
- \( x_i \) : Expected number of out-of-stocks at retailer \( i \), per period
- \( c(x_i) \) : Penalty cost function for retailer \( i \) (function of number of out-of stock items per period)
- \( y_i \) : Expected inventory position at the end of a period at retailer \( i \)
- \( h(y_i) \) : Holding cost at retailer \( i \) per unit period (function of inventory position at the end of a period)
- \( h_i \) : Holding cost per unit per period
- \( C_i \) : Selling price of a unit at retailer \( i \)
- \( c_i \) : Penalty cost fraction of \( C_i \)
- \( S_i \) : Echelon Inventory position at the beginning of a period at retailer \( i \)
- \( d_i \) : Demand at retailer \( i \) during the period
- \( z_i \) : The allocated inventory to retailer \( i \) from the supplier for the period
- \( Z \) : Row matrix representing allocation of each retailer
- \( Z_0 \) : Initial allocation matrix
- \( f(d_i) \) : Probability density (mass) function of the demand distribution at retailer \( i \)
- \( TC_i \) : Total holding and penalty cost for retailer \( i \)
- \( TC_s \) : Total system cost
- \( N \) : Number of retailers
- \( SL_i \) : % Service Level at retailer \( i \)
- \( SLR_i \) : % Service level target of retailer \( i \)
- \( SS_i \) : Safety Stock at retailer \( i \) to deal with demand during lead times
- \( R \) : Assessment Period

**Development of Model**

Following equations can be used to estimate the expected safety stock, service level, holding cost, penalty cost, and total cost of the system.

Safety Stock Required at retailer \( i \):  
\[
SS_i = \frac{L_i}{R} \sum_{i=0}^{\infty} f(d_i) * d_i \tag{1}
\]

Service Level at retailer \( i \) (Non-stock-out probability),  
\[
SL_i = \left(1 - \frac{x_i}{z_i + S_i - SS_j}\right) * 100 \tag{2}
\]

Expected cost of out-of-stock at retailer \( i \):  
\[
c(x_i) = c_i * C_i * x_i = c_i * C_i * \sum_{d_i=z_i+s_i+1}^{\infty} [d_i - (z_i + s_i)] * f(d_i) \]

Expected holding cost at retailer \( i \):  
\[
h(y_i) = h_i * y_i = h_i * \sum_{d_i=0}^{z_i+s_i-1} [(z_i + s_i) - d_i] * f(d_i) \tag{4}
\]

The expected total cost for retailer \( i \):  
\[
TC_i = c_i * C_i * \sum_{d_i=z_i+s_i+1}^{\infty} [d_i - (z_i + s_i)] * f(d_i) + h_i * \sum_{d_i=0}^{z_i+s_i-1} [(z_i + s_i) - d_i] * f(d_i) \tag{5}
\]
Since the available stock with the supplier has been completely allocated to the retailers, total cost of the system is the sum of all penalty and holding costs for all retailers.

\[ TC_s = \sum_{i=1}^{N} TC_i = \sum_{i=1}^{N} c_i \cdot C_i \cdot \left( \sum_{d_i=z_i+s_i}^{\infty} \left[ d_i - (z_i + s_i) \right] \cdot f(d_i) + h_i \cdot \sum_{d_i=0}^{z_i+s_i-1} \left( z_i + s_i - d_i \right) \cdot f(d_i) \right) \]  

This is the objective function of the allocation problem. The total cost function of the system is to be minimized under the constraint of the available inventory with the supplier. That means, 

\[ \text{Minimize:} \]

\[ TC_s = \sum_{i=1}^{N} c_i \cdot C_i \cdot \left( \sum_{d_i=z_i+s_i}^{\infty} \left[ d_i - (z_i + s_i) \right] \cdot f(d_i) + h_i \cdot \sum_{d_i=0}^{z_i+s_i-1} \left( z_i + s_i - d_i \right) \cdot f(d_i) \right) \]  

\[ \text{Subject to:} \sum_{i=1}^{N} z_i \leq A \quad \text{SL}_i = \text{SLR}_i \quad \text{for all i} \]  

It can be seen from the above equations that the objective function will consists of very large (infinite) number of terms. Therefore, it will not be feasible to find out an exact analytical solution to this problem. We propose an iterative procedure to solve this problem. The advantage of this procedure is that it can handle any type of discrete demand distribution and cost functions: linear as well as non-linear. Following algorithm is developed for this the iterative allocation and rationing:

10 \hspace{1cm} j=1 \quad \text{Calculate TCmin} = \text{TC}_s \; (\text{for initial allocation } Z_0)
20 \hspace{1cm} i=1
30 \{ \begin{align*}
& \text{allocate } z_i = z_i + 1; \\
& \text{calculate } \text{TC}_s \text{ with the present matrix } Z; \\
& \text{assign } \text{TC}_s(i) = \text{TC}_s; \\
& \text{remove the allocation } z_i = z_i - 1; \\
& \quad i = i + 1
\end{align*} \}

\begin{align*}
& \text{IF (i < N) GO TO 30} \\
& \text{ELSE Find MIN [TC}_s(i); i = 1..N]; \text{assign this to } \text{TC}_s(\text{imin}) \\
& \text{Find } \text{imin} \text{ for which } \text{TC}_s(i) \text{ is minimum} \\
& \text{IF (TC}_s(\text{imin}) \leq \text{TCmin}) \\
& \quad \{ \text{TCmin} = \text{TC}_s(\text{imin}) \\
& \quad \text{Confirm the allocation for } z_{imin} = z_i + 1 \\
& \quad \text{Update the matrix } Z_i \text{ by including this change} \\
& \quad \text{Calculate } SL_i = \left( 1 - \frac{x_i}{z_{imin} + S_i - SS_i} \right) \cdot 100 \\
& \text{IF SL_i \geq \text{SLR}_i \text{ Exclude } z_{imin} \text{ from the set of i from further iterations} } \\
& \text{IF } \sum_{i=1}^{N} z_i < A \\
& \quad \{j = j + 1 \text{ GO TO 20}\} \\
& \text{ELSE } Z_i \text{ of the previous iteration is optimum } \}
\end{align*}

5. Numerical Example:
Consider a supply chain with one supplier and 4 retailers. A central decision maker distributes the available inventory of the supplier to the four retailers every week. The objective is to minimize the total holding and stock-out costs of all locations. Following data are available to the central decision maker:
Table 1: Data for Numerical Example

<table>
<thead>
<tr>
<th>Retailers</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Poisson Demand per week</td>
<td>5</td>
<td>8</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>Lead Times (Hours)</td>
<td>36</td>
<td>20</td>
<td>32</td>
<td>42</td>
</tr>
<tr>
<td>Holding Cost per unit per week ($)</td>
<td>6.5</td>
<td>7.8</td>
<td>5.7</td>
<td>4.9</td>
</tr>
<tr>
<td>Stock-out Penalty Cost per unit per week ($)</td>
<td>1.25</td>
<td>2.30</td>
<td>3.20</td>
<td>1.80</td>
</tr>
<tr>
<td>Maximum Retail Price of a unit ($)</td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>Inventory Position at Starting of the period</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

The available inventory is to be allocated or rationed optimally among the four retailers under the following situations:

**Case 1:** Available stock with supplier, \( A = 31 \) units

**Case 2:** Unlimited stock is available with the supplier

**Solution: Case 1: Limited Stock at Supplier**

The constraint in this case is the number of units available with the supplier for rationing.

Mathematically,

\[
\sum_{i=1}^{4} z_i \leq 31
\]

Let us assume the strategy that 60% of demand of each retailer is allocated initially. Based on this, we get the following starting matrix, corresponding total cost, and the total units allocated:

Starting Solution: \( Z_0 = \{3 \ 5 \ 6 \ 8 \} \)

\[
TC_s(0) = $181.60 \quad \text{and} \quad \sum_{i=1}^{4} z_i = 22
\]

After 9 iterations, we get the following optimal allocation: \( Z_{10} = \{4 \ 7 \ 9 \ 11\} \)

\[
TC_s(10) = $119.74 \quad \text{and} \quad \sum_{i=1}^{4} z_i = 31
\]

Expected Service Level (non-stock-out probability) of this optimal rationing at each retailer are:

\( SL_1 = 90.13\% \), \( SL_2 = 95.27\% \), \( SL_3 = 97.07\% \), and \( SL_4 = 98.10\% \) respectively.

**Case 2: Unlimited Stock at Supplier**

In this case, the supplier has unlimited number of units available for allocation.

As in case 1, let us assume the initial solution matrix \( Z_0 = \{3 \ 5 \ 6 \ 8 \} \)

After 12 iterations from this position, we get:

\( Z_{12} = \{5 \ 7 \ 10 \ 12\} \)

\[
TC_s(12) = $116.96 \quad \text{and} \quad \sum_{i=1}^{4} z_i = 34
\]

The 13th iteration from starting position, we get:

\( Z_{13} = \{5 \ 8 \ 10 \ 12\} \)

\[
TC_s(13) = $117.79 \quad \text{and} \quad \sum_{i=1}^{4} z_i = 35
\]

Since \( TC_s(13) > TC_s(12) \), \( Z_{12} \) is the optimal allocation in this case. This shows that even though the supplier has unlimited number of units available, it is better to allocate only 34 units among the four retailers to minimize the total cost. Service levels corresponding to this optimal allocation at each retailer are:

\( SL_1 = 95.74\% \), \( SL_2 = 95.27\% \), \( SL_3 = 98.44\% \), and \( SL_4 = 98.96\% \)

6. Conclusions

The proposed algorithm is capable of finding cost-optimal solutions for rationing and allocation problems under the constraints of limited or unlimited stock availability of the supplier and unique service level of each retailer. The algorithm assumes an initial solution matrix and subsequently uses an iterative procedure. Selection of initial solution matrix has no effect on the optimal solution, if at least one unit is allocated by the subsequent iterations to each retailer. Strategies proposed and tested in the
paper for deciding the initial starting solution, can be effectively used for reducing the number of iterations required to reach the final solution. The proposed algorithm has several advantages over existing models. First of all it ensures optimal allocation. This is because, starting from the initial solution matrix, each iteration improves upon the previous one and converging to a minimum cost allocation. The algorithm can be used even if each retailer has a different demand distribution. Allocation and rationing problems of two-stage supply chains with any number of retailers can be handled. The algorithm ensures an exact, unbiased cost optimal solution. It is also accurate, easy to implement and communicate.

7. References